Scalar hairy black holes and solitons in a gravitating Goldstone model

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Abstract

We study black hole solutions of Einstein gravity coupled to a specific global symmetry breaking Goldstone model described by an $O(3)$ isovector scalar field in four spacetime dimensions. Our configurations are static and spherically symmetric, approaching at infinity a Minkowski spacetime background. A set of globally regular, particle-like solutions are found in the limit of vanishing event horizon radius. These configurations can be viewed as *'regularised'* global monopoles, since their mass is finite and the spacetime geometry has no deficit angle. As an unusual feature, we notice the existence of extremal black holes in this model defined in terms of gravity and scalar fields only.

1 Introduction

Black holes with scalar hair are rather a common presence in the lanscape of gravity solutions with anti-de Sitter asymptotics. Some of these configurations have found interesting applications in the context of gravity/gauge duality, see *e.g.* [1]. The situation is, however, rather different in the absence of a cosmological constant. Asymptotically flat black holes in models featuring scalar fields are rather scarce. Such solutions have been mainly studied as counterexamples to the no hair conjecture [2] and typically contain also gauge fields (for a review of this topic, see [3]). Interestingly, as proven in the case of Einstein-Skyrme theory, there are hairy black hole solutions even in theories with scalar fields only [4]. These Einstein-Skyrme solutions were shown to be stable [5], [6].

Perhaps the simplest examples of black hole solutions in a model with scalar fields only are found in a symmetry breaking model featuring an $O(3)$ scalar isovector field. These are the black holes inside the global monopoles [7], which were discussed in [8], [9]. Global monopoles are topological defects that arise in certain theories where a global symmetry is spontaneously broken. Like the well-known 't Hooft-Polyakov monopoles, these configurations are constructed within an hedgehog-type Higgs field Ansatz and possess a unit conserved topological charge, which is the winding number of the scalars. However, for both solutions with a regular origin and black holes, the energy density decays as $1/r^2$ at large distances and hence their masses, defined in the usual way, diverge. This leads to a deficit solid angle in the geometry of the space and the resulting spacetime is not strictly asymptotically flat[1].

Hairy black holes with finite mass approaching at infinity a Minkowski spacetime background are found in the gauged generalisation of this model, with a non-Abelian (local) gauge group $SO(3)$. This, of course, is the usual Georgi-Glashow model supporting 't Hooft-Polyakov monopoles. These configurations were extensively studied in the literature, from various directions [12]. In contrast to the ungauged version, the black holes inside 't Hooft-Polyakov monopoles become extremal in a critical limit. Heuristically, this property can be attributed to the existence of a magnetic charge in this model. As is well known, this charge is completely specified in terms of the scalar triplet of Higgs fields [13].

This leads to the interesting question, as to whether one can find finite mass solutions with similar properties in a simple model with a scalar isovector field only, *i.e.* without gauge fields. Such solutions would still possess a 'magnetic'-type topological charge and thus may allow for extremality; however, their existence would require a deviation from the standard scalar fields action. In this work we answer this question by constructing solutions of a specific Goldstone model in 3+1 dimensions, originally proposed in [14]. Our solutions have finite mass and approach a Minkowski spacetime background at infinity. They also share a number of basic properties with the gravitating 't Hooft-Polyakov monopoles. In particular, and in strong contrast with the usual global monopoles with an event horizon in [8], [9], we find that extremal black holes exist even in a model with scalar fields only.

2 The model

2.1 Action and field equations

We consider the 3+1 dimensional action

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} - L_m\right), \tag{1}$$

where the gravity part of the action is the usual Einstein-Hilbert action with curvature scalar R and G the gravitational coupling constant. L_m is the Lagrangian of the matter fields, which is given by a symmetry breaking model in 3+1 spacetime dimension to which

[1]See Ref. [10] for a discussion of this type of asymptotics together with a definition of the mass through the application of the standard Hamiltonian analysis. Global monopoles and black holes in Einstein-Goldstone model with a cosmological constant are studied *e.g.* in [11]. The mass and action of the solutions are also computed there by using a boundary counterterm subtraction method.

we refer as a Goldstone model. In general, L_m contains three different parts,

$$L_m = \lambda_1 U(|\vec{\phi}|)\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} + \lambda_2(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 + V(|\vec{\phi}|), \tag{2}$$

with $\vec{\phi} \equiv \phi^a = (\phi^1, \phi^2, \phi^3)$ a triplet of real scalar fields. The first part above is the usual kinetic term multiplied with a correction factor U which depends only on the magnitude of $\vec{\phi}$; the second part $(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2$ is a Skyrme-like term, while $V(|\vec{\phi}|)$ is a symmetry breaking potential.

The corresponding Einstein equations are found by varying (1) with respect to $g_{\mu\nu}$ and read

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{3}$$

with the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} \;\; = \;\; & 2\,\lambda_1\, U(|\vec{\phi}|)\left[(\partial_\mu\phi^a\partial_\nu\phi^a) - \frac{1}{2}\,g_{\mu\nu}\,(\partial_\tau\phi^a\partial_\lambda\phi^a)\,g^{\tau\lambda}\right] - g_{\mu\nu}\,V(|\vec{\phi}|) \\ & +4\lambda_2\left[\left(\partial_{[\mu}\phi^a\partial_{\tau]}\phi^b\right)\,\left(\partial_{[\nu}\phi^a\partial_{\lambda]}\phi^b\right)\,g^{\tau\lambda} - \frac{1}{4}\,g_{\mu\nu}\,\left(\partial_{[\rho}\phi^a\partial_{\tau]}\phi^b\right)\,\left(\partial_{[\sigma}\phi^a\partial_{\lambda]}\phi^b\right)\,g^{\rho\sigma}\,g^{\tau\lambda}\right] \end{aligned} \tag{4}$$

The equation of motion for the scalar fields is

$$\begin{aligned} \lambda_1 & \left[\frac{2}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,U(|\vec{\phi}|)\partial^\mu\phi^a\right) - (\partial_\mu\phi^b\partial^\mu\phi^b)\partial_{\phi^a}U(|\vec{\phi}|)\right] \\ & + \frac{4\lambda_2}{\sqrt{-g}}\,\partial^\nu\,\left(\sqrt{-g}\,\partial_{[\mu}\phi^a\,\partial_{\nu]}\phi^b\right) - \,\partial_{\phi^a}V(|\vec{\phi}|) = 0\;. \end{aligned} \tag{5}$$

As a general feature, the scalar field $\vec{\phi}$ is a relic of a Higgs field and has the same dimensions L^{-1} as a gauge connection. Asymptotically, is satisfies the symmetry breaking boundary condition

$$\lim_{r\to\infty}|\vec{\phi}| = \eta. \tag{6}$$

2.2 The spherically symmetric ansatz

In this paper we shall consider spherically symmetric globally regular and black hole solutions to the system (1). A generally enough metric ansatz reads

$$ds^2 = -f_0(r)dt^2 + f_1(r)dr^2 + f_2(r)(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{7}$$

where t is the time coordinate, r is the radial coordinate (with $r^2 = x^a x^a$), while θ and φ are the angular coordinates within the usual range. In the numerical construction of asymptotically flat configurations, we have mainly employed the usual Schwarschild coordinates with

$$f_0(r) = N(r)\sigma(r)^2, \quad f_1(r) = \frac{1}{N(r)}, \quad f_2(r) = r^2, \quad \text{and} \quad N(r) = 1 - \frac{2m(r)}{r}, \tag{8}$$

where $m(r)$ may be interpreted as the total mass-energy within the radius r. For black hole solutions, the event horizon is at $r = r_h$ where $N(r_h) = 0$ and $\sigma(r_h) > 0$. For solitons, $r = 0$ is a regular origin, with $N(0) = 1$, $\sigma(0) > 0$.

For the scalar field, we use the usual hedgehog ansatz, with

$$\phi^a = \eta\, h(r)\, \hat{x}^a\,, \tag{9}$$

and $\hat{x}^a = x^a/r = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

With this ansatz, the field equations (3), (5) take the relatively simple form (where the prime denotes derivative with respect to r):

$$m' = \alpha^2\left(Nh'^2 T_1 + T_2\right), \quad \frac{\sigma'}{\sigma} = \frac{2\alpha^2}{r} h'^2 T_1, \tag{10}$$

for the metric functions, and

$$(\sigma N h' T_1)' = \frac{1}{2}\sigma\left(Nh'^2\frac{\partial T_1}{\partial h} + \frac{\partial T_2}{\partial h}\right), \tag{11}$$

for the scalar amplitude. In these relations we define as usual

$$\alpha^2 = 4\pi G\eta^2, \tag{12}$$

and we use the shorthand notation

$$T_1 = \lambda_1 U(h) r^2 + 2\lambda_2 h^2, \quad T_2 = 2\lambda_1 U(h) h^2 + \lambda_2\frac{h^4}{r^2} + r^2 V(h). \tag{13}$$

As originally discussed in [13], the possibility of a nonvanishing 'magnetic' charge in a model with an $O(3)$ isovector scalar field is determined by the homotopy class of the scalar fields only, being independent of the gauge fields. Moreover, this is true no matter what action principle determines the dynamics of ϕ^a. Following [13], one can define a 't Hooft 'electromagnetic' tensor

$$F_{\mu\nu} = -\epsilon_{abc}\hat{\phi}^a\partial_\mu\hat{\phi}^b\partial_\nu\hat{\phi}^c, \tag{14}$$

where $\hat{\phi}^a = \phi^a/|\vec{\phi}|$. A straightforward computation shows that for the hedgehog ansatz (9), the only nonvanishing component of this tensor is $F_{\theta\varphi} = \cos\theta$, which after integration over S^2 gives a unit 'magnetic' charge for the solutions, as expected. Note that this is a generic feature independent on the coupling with gravity, or, on the existence of finite energy solutions of the equations (10), (11).

2.3 A virial identity

By expressing the curvature scalar R in terms of the metric function $m(r)$ and $\sigma(r)$ and dropping a total divergence term, we obtain the following form of the effective Lagrangean of our static spherically symmetric system:

$$L_{eff} = \sigma\left(m' - \alpha^2(Nh'^2 T_1 + T_2)\right)\,. \tag{15}$$

This form of the reduced Lagrangean allow us to obtain an interesting virial relation. Following the approach proposed in [15], let us assume the existence of a solution $m(r), \sigma(r), h(r)$ with suitable boundary conditions at the horizon and at infinity. Then each member of the 1-parameter family $F_\lambda(r) \equiv F(r_h + \lambda(r - r_h))$ (with $F = (m, \sigma, h)$ and λ some arbitrary real parameter (which should not be confused with the constants $\lambda_1,\ \lambda_2$)) assumes the same boundary values at $r = r_h$ and $r = \infty$. Then the action $S[m_\lambda, \sigma_\lambda, h_\lambda]$ must have a critical point at $\lambda = 1$, $[dS/d\lambda]|_{\lambda=1} = 0$. The result is the following virial relation

$$\int_{r_h}^{\infty} dr(\mathcal{P}_0 + \lambda_1 \mathcal{P}_1 + \lambda_2 \mathcal{P}_2) = 0, \tag{16}$$

where

$$\mathcal{P}_0 = \sigma r^2 V(h)(3 - \frac{2r_h}{r}), \quad \mathcal{P}_1 = \sigma U\Big(r^2 h'^2 (1 - \frac{2r_h}{r}(1 - \frac{m}{r}) + 2h^2\Big),$$
$$\mathcal{P}_2 = \sigma\Big(2h^2 h'^2\big(-1 + \frac{2m}{r}(2 - \frac{r_h}{r})\big) - \frac{h^4}{r^2}(1 - \frac{2r_h}{r})\Big), \tag{17}$$

Setting $r_h = 0$ in the above relations leads to a virial identity valid for gravitating particle-like solutions. Furthermore, a virial relation for the nongravitating limit of this model (*i.e.* solutions of (11) in a fixed Minkowski spacetime background, no backreaction) is found by taking $m = 0,\ r_h = 0$ in (16), (17).

2.4 The global monopoles

For $U(|\vec{\phi}|) = 1$, $\lambda_2 = 0$, the action (1) corresponds to the usual global monopole model, with

$$L_m = \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} + \lambda(|\vec{\phi}| - \eta^2)^2, \tag{18}$$

whose solutions were extensively studied in the literature. The configurations with a regular origin are found with a usual shooting method by which one adjusts the value of $h'(0)$, integrates outward to large radius, and shoots for an asymptotic boundary condition such that $h(r \to \infty) \to 1$. Such solutions represent global monopoles of unit charge and are parametrized by $h'(0)$. It turns out that above a critical value $\alpha = \alpha_{max}$ no such solutions can be found [8], [16]. As α is increased towards the critical α_{max}, the value of $h'(0)$ for which the static monopole solution is found decreases toward zero. The critical solution represents the point at which the static monopole becomes identical to de Sitter space in which $h(r) = 0$ and the symmetry-breaking potential reduces to a cosmological constant [17]. The picture for black hole sitting inside global monopoles is more complicated [9], [18]. For values of α below some critical value, one finds black holes with arbitrarily large radius. Above this critical value, the branch of black holes bifurcates with the Schwarzschild-de Sitter solution.

However, a generic feature of all these solutions is that the kinetic term $\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}$ does not vanish at infinity, which leads to a divergent total mass[2]. The absence of finite mass solutions can also be seen from the virial identity (16), since one can show that both $\mathcal{P}_0$ and $\mathcal{P}_1$ are strictly positive quantities.

2.5 The new model: *'regularised'* global monopoles

To cure this mass divergence, we consider in this work a nontrivial correction factor in front of the $\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}$ term,

$$U(|\vec{\phi}|) = (|\vec{\phi}| - \eta^2)^2, \tag{19}$$

where $|\vec{\phi}| \to \eta$ at infinity. This expression of U regularises the contribution of the kinetic term to the mass-energy. However, the virial identity (16) forbids again the existence of finite mass solutions unless the Skyrme-like term is also included, $\lambda_2 > 0$. Technically, this is a consequence of the fact that the sign of $\mathcal{P}_2$ in (16) is not fixed, and, in fact it becomes negative for large values of r. Heuristically, similar to the Hopf or Skyrme models [19], the quartic term $(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2$ provides the extra repealing force allowing for finite mass to the solutions.

Also, by using the field equation for $\vec{\phi}$, one can show that no finite mass solutions are found in a truncated model with the Skyrme-like term only[3]. Therefore, we are forced to consider one more term in (2) in addition to $(\partial_\mu\vec{\phi}\times\partial^\nu\vec{\phi})^2$. This can be the kinetic term $\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}$ with an extra factor given by (19) and/or a symmetry-breaking potential term. We have verified the existence of finite mass solutions of the equations (10) in a model with

$$L_m = (\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})^2 + \lambda(|\vec{\phi}| - \eta^2)^2, \tag{20}$$

i.e. with a quartic term only. However, we have found more interesting to keep all terms in (2), and to consider the general model with a correction factor $U(|\vec{\phi}|)$ given by (19).

One should remark that although it may look unusual, the model (2) (with $U(|\vec{\phi}|)$ given by (19), $V(|\vec{\phi}|) = \lambda(|\vec{\phi}| - \eta^2)^4$ and $\lambda_1 = 4\lambda_2 = 1$) has some higher dimensional origin. As discussed in [14], the Lagrangian (2), considered on flat $\mathbb{R}^3$, arises from the gauge decoupling limit (*i.e.* no Yang-Mills fields) of the three dimensional $SO(3)$ gauged Higgs model descended from the *second* ($p = 2$) member of the Yang-Mills hierarchy on $\mathbb{R}_3 \times S^5$. The resulting global symmetry breaking Goldstone model considered in a fixed four dimensional Minkowski spacetime background admits finite energy solutions [21] (see also Ref. [22] for higher dimensional generalisations). In this case, it is straightforward to show that the corresponding energy density is bounded from below by

$$\varrho = \frac{1}{4\pi}\varepsilon_{ijk}\varepsilon^{abc}\left(\eta^2 - |\vec{\phi}|^2\right)\phi^a_i\,\phi^b_j\,\phi^c_k = \frac{1}{4\pi}\varepsilon_{ijk}\varepsilon^{abc}\,\partial_i\left[\left(\eta^2 - \frac{3}{5}|\vec{\phi}|^2\right)\phi^a\,\phi^b_j\,\phi^c_k\right], \tag{21}$$

[2]One can see from (10) that since $h \to 1$ for large enough r, then $m' \sim 2\alpha^2$ for solutions with $U = 1$.

[3]To this end, one writes the equation for h as $(\sigma N h' h^3)' = \sigma(2Nh'^2h^2 + h^4/r^2)$. Then by integrating the origin/event horizon to infinity, one can show that $h \equiv 0$.

(with $i, j = 1, 2, 3$). Then, as discussed in [21], the total mass of the flat space solutions has a lower bound (which is never saturated), $M \geq Q$, with $Q = \int d^3x \ \varrho = 4/5$ a topological charge for our spherically symmetric configurations[4]. The locally gauged version of this model corresponds to a specific Yang-Mills–Higgs theory, whose solutions were studied quantitatively in [20].

However, for the purposes of this work, we are interested in this model mainly because it provides a simple toy model admiting finite mass, black hole solutions with a symmetry breaking scalar field outside the horizon, carying also a topological charge.

2.6 Boundary values and asymptotic behaviour

We start by noticing that by using a suitable redefinition of λ_1, λ_2 together with a rescaling of the radial coordinate, one can always take $\lambda_1 = 4\lambda_2 = 1$ without any loss of generality. Also, to simplify the general picture, we set $V(|\vec{\phi}|) = 0$ in what follows (although we could confirm that finite mass solutions exist also for a nonvanishing scalar potential). This leaves us with a single essential parameter of the problem, α, which, for a given G, is fixed by the *v.e.v.* of the Goldstone field.

The asymptotic form of the functions m, σ, h can be systematically constructed in both regions, near the event horizon/origin and for $r \to \infty$. The nonextremal solutions possess the following expansion near the event horizon:

$$\begin{aligned} m(r) &= \frac{r_h}{2} + m_1(r - r_h) + O(r - r_h)^2, \quad \sigma(r) = \sigma_h + \sigma_1(r - r_h) + O(r - r_h)^2, \\ h(r) &= h_0 + h_1(r - r_h) + O(r - r_h)^2. \end{aligned} \tag{22}$$

For a given event horizon radius r_h, the essential parameters characterizing the event horizon are h_0 and $\sigma_h > 0$, which fix all higher order coefficients in (22). (These constants are related in a complicated way to the parameters M, c_1 of the solutions in the far field expansion (27).) One finds *e.g.* for the lowest order terms

$$\begin{aligned} m_1 &= \frac{\alpha^2 h_0^2}{r_h^2}\left(h_0^2 + 2(1 - h_0^2)^2 r_h^2\right), \quad h_1 = \frac{4h_0}{2(1 - 2m_1)r_h}\frac{\left((1 - h_0^2)^2 r_h^2 + h_0^2(1 - 2(1 - h_0^2)r_h^2)\right)}{\left(2h_0^2 + (1 - h_0^2)^2 r_h^2\right)}, \\ \sigma_1 &= 2\alpha^2\sigma_h \frac{h_1^2}{r_h}\left(2h_0^2 + (1 - h_0^2)^2 r_h^2\right). \end{aligned} \tag{23}$$

The Hawking temperature and the entropy of the black holes are given by

$$T_H = \frac{1}{4\pi}\sigma(r_h)N'(r_h), \quad S = \frac{A_H}{4G}, \quad \text{with} \quad N'(r_h) = \frac{1}{r_h}(1 - 2m'(r_h)) \quad \text{and} \quad A_H = 4\pi r_h^2. \tag{24}$$

As we shall see, in the limit of zero event horizon radius of the black holes, the solutions describe particle-like globally regular solitons. $r = 0$ is in this case a regular origin, the

[4] The model (18) (with $V(|\vec{\phi}|) = 0$) can also be thought as resulting from the gauge decoupling limit of the *first* member of the Yang-Mills hierarchy on $\mathbb{R}_3 \times S^1$. However, no lower bound for their action (similar to that resulting from (21)) is found in that case.

corresponding approximate solution close to that point being

$$h(r) = \bar{h}_1 r + h_3 r^3 + O(r^5), \quad m(r) = \alpha^2 \bar{h}_1^2 (1 + \bar{h}_1^2) r^3 + O(r^5), \qquad (25)$$
$$\sigma(r) = \sigma_0 + \alpha^2 \sigma_0 \bar{h}_1^2 (1 + 2\bar{h}_1^2) r^2 + O(r^4),$$

where

$$h_3 = \frac{\bar{h}_1^3 \left(-1 + \alpha^2 (3 + 6\bar{h}_1^2 + 2\bar{h}_1^4)\right)}{5(1 + 2\bar{h}_1^2)}, \qquad (26)$$

with two free parameters $\bar{h}_1 = h'(0)$ and σ_0.

We assume that the spacetime is asymptotically flat, which leads to the following expansion as $r \to \infty$

$$h(r) = 1 + ce^{-2r} - \frac{1}{4r^2} + \dots, \; m(r) = M - \frac{\alpha^2}{4r} + \dots, \quad \sigma(r) = 1 - \frac{\alpha^2}{6r^6} + \dots \; , \qquad (27)$$

in terms of two free parameters M, c_1, with M the mass of the solutions.

3 Numerical results

3.1 The solitons

Since the black holes of our model smoothly emerge from the particle-like solutions, we start by discussing this limit.

The solitons are the gravitating generalisation of the flat space solutions considered in [21]. This limit is recovered for $\alpha = 0$, $N(r) = \sigma(r) = 1$. The backreaction is included by slowly increasing the value of α. For a given α, the solutions may exist for discrete values of $h'(0)$ (which is the shooting parameter of the problem).

We follow the usual approach and, by using a standard ordinary differential equation solver[5], we evaluate the initial conditions (25) at $r = 10^{-6}$, adjusting for the fixed shooting parameter and integrating towards $r \to \infty$. The integration stops when the asymptotic limit (27) is reached with reasonable accuracy.

Although solutions with nodes in $h(r)$ do also exist, we discuss in what follows the configurations with a monotonic behaviour of the scalar function only. As one can see from the field equations (10), the metric functions m, σ are also strictly increasing with r. The Figure 1 shows the profiles of three solutions with different values of α.

As expected, the solutions exist for a finite range of α, with $\alpha < \alpha_{max} \simeq 1.545$. The picture can be summarize as follows: when α increases, the dimensionless mass

[5] Some of the solutions were also constructed by using an isotropic coordinate system, with $f_2 = f_1 r^2$ in the general ansatz (7). In that case we have employed a different solver [23] which involves a Newton-Raphson method for boundary value ordinary differential equations. This approach also allows us to extend our consideration to a more general case of the solutions with axial symmetry and higher values of the winding number (this study will be reported elsewhere).

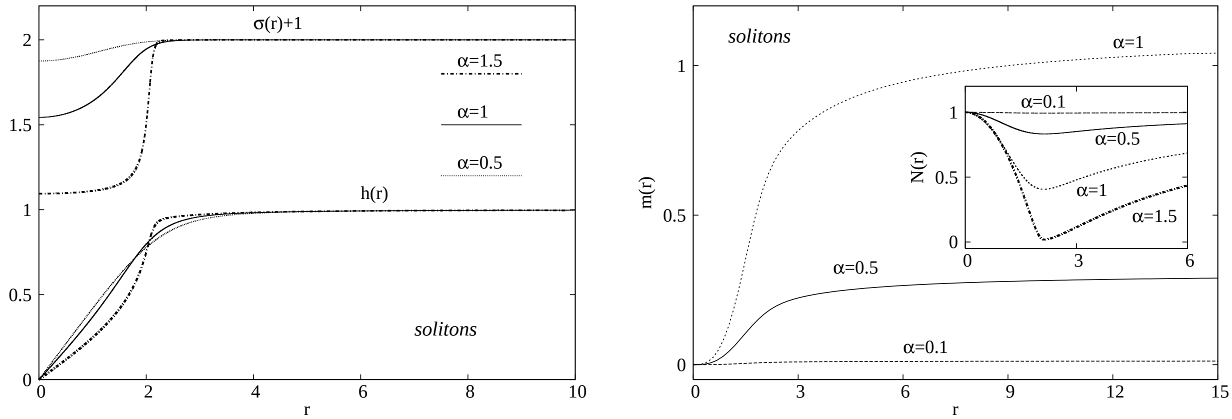

Figure 1. The profiles of scalar soliton solutions are shown for several different values of the coupling constant α.

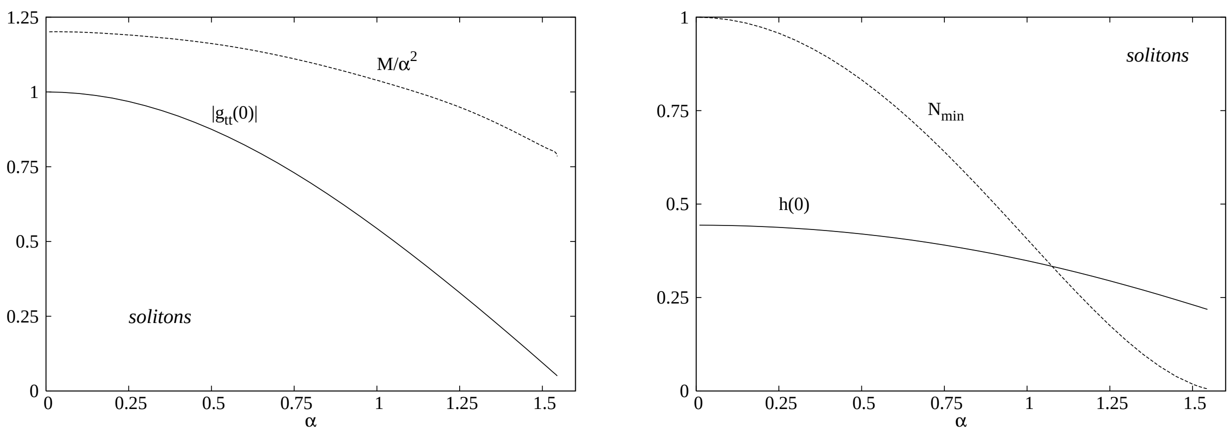

Figure 2. Some features of the scalar soliton solutions are shown as a function of the coupling parameter α.

parameter M/α^2 decreases, as well as the value $\sigma(0)$. The minimum N_m of the function $N(r)$ also decreases, as indicated in Figures 1,2. This minimum becomes more pronounced for larger α, and finally, a horizon is found for $\alpha = \alpha_{max}$ and some finite value of $r = r_c \simeq 2.15$.

We have noticed that, within the numerical errors, $N'(r_c) = 0$, *i.e.* the function $N(r)$ has a double zero at $r = r_c$. Also, the proper distance $\ell = \int_0^r dr/\sqrt{N(r)}$ diverges for $r \to r_c$. As a result, the spatial geometry on the hypersurface $t = const.$ develops an infinite throat separating the interior region with a smooth origin and non-trivial scalar field, from the exterior region which corresponds to a finite mass, extremal black hole with scalar hair.

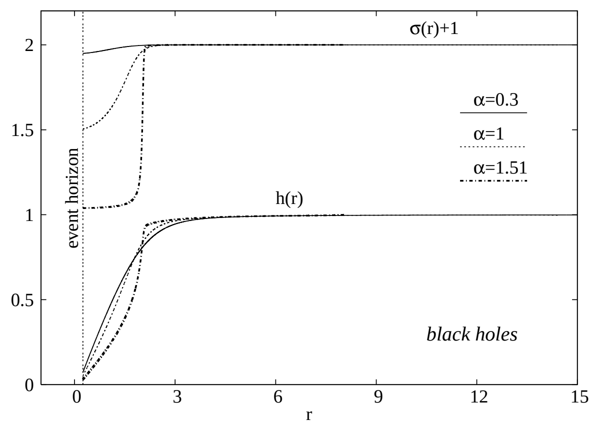

Figure 3. The profiles of typical black hole solutions with different values of α and the same event horizon radius are shown as a function of the radial coordinate.

It may be interesting to remark that this strongly contrasts with the picture found in [17] for the global monopoles. There, a de Sitter spacetime is approached for a maximal value of α, with $h'(0) \to 0$ in that limit. However, for the model under consideration in this work, the picture is qualitatively similar to that valid for gravitating 't Hooft-Polyakov monopoles [12]. In that case, the metric in the exterior region is that of an extremal Reissner-Nordstöm black hole. The near horizon geometry of the exterior configuration here is described by the exact solution (29) with $h_0 \simeq 0.92$.

3.2 The black holes

According to the standard arguments, one can expect black hole generalisations of the regular configurations to exist at least for small values of the horizon radius r_h. This is indeed confirmed by the numerical analysis. In our approach, we have restricted our integration to the (physically more relevant) region outside of the horizon, $r \geq r_h$. Given (r_h, α), the black hole solutions may exist for a set of discrete values of the scalar field on the horizon, h_0. Similar to the soliton case, we restrict our study to solutions with no nodes in $h(r)$. The profiles of several solutions with the same event horizon radius are shown in Figure 3, for several values of α.

Starting from a regular solution with a given α and increasing the event horizon radius, we find a branch of black hole solutions. For $r_h \ll 1$, the solutions resemble small black holes sitting in the center of the regular lumps, the latter being almost unaffected for $r \gg r_h$ by the presence of the black hole. The Hawking temperature decreases along this branch while the mass and the value of the scalar field on the horizon increase.

The issue of the limiting behaviour of this branch of solutions is more complicated and depends crucially on the value of α. For $\alpha < \alpha_c = 3\sqrt{3}/4$, we could construct solutions with very large value of r_h and they are likely to exist for arbitrary values of the event horizon radius. The value at the horizon of the scalar field increases with r_h, approaching

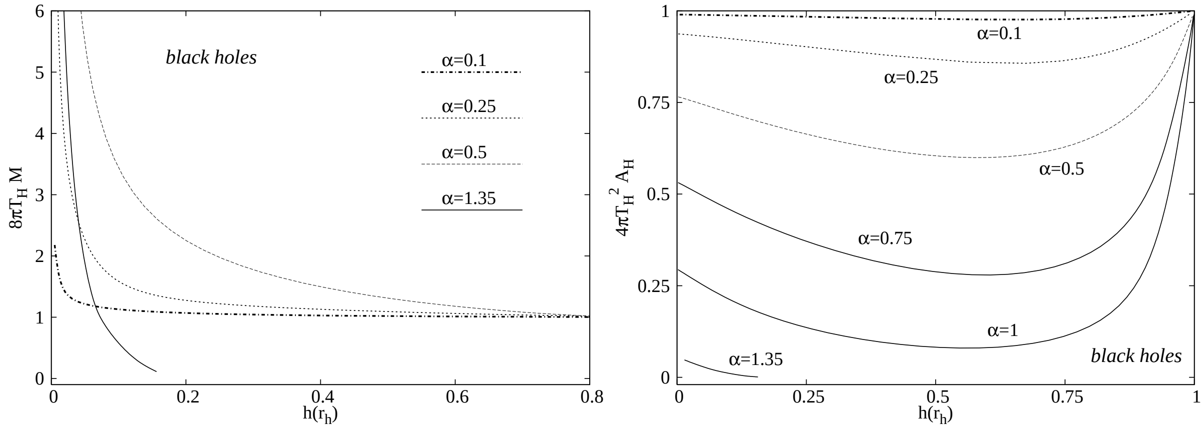

Figure 4. The (suitably normalized-) dimensionless quantities $T_H M$ and $T_H^2 A_H$ are shown as functions of the scalar field on the horizon $h(r_h)$, for several fixed values of the parameter α. The control parameter here is the event horizon radius r_h, with $h(r_h) = 0$ and $h(r_h) = 1$ corresponding to the soliton and the Schwarzschild black hole limits, respectively.

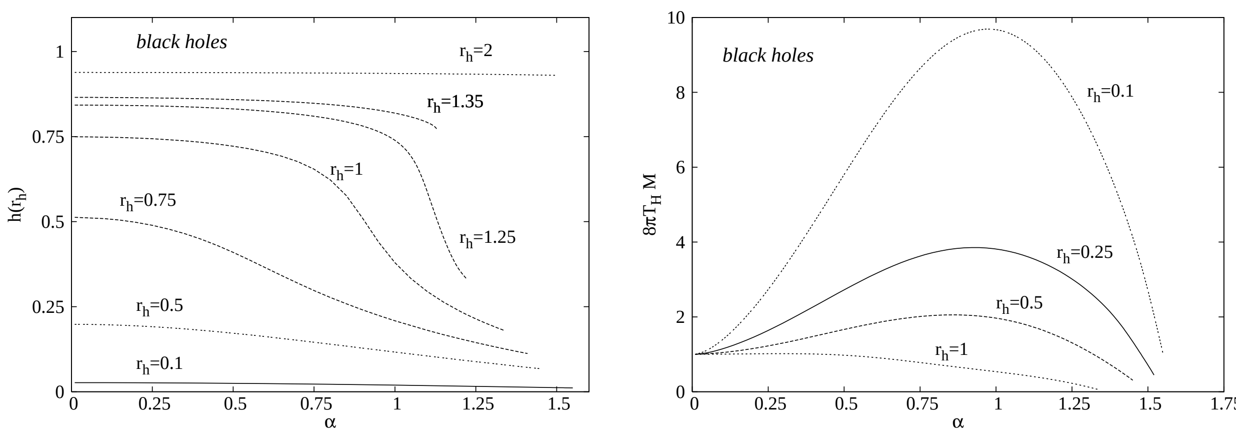

Figure 5. Some features of the black hole solutions are shown as functions of α, for several fixed values of the event horizon radius.

asymptotically the unit value. As seen in Figure 4, the large black holes are essentially Schwarzschild solutions, with a small scalar field outside the horizon. The intrinsic parameter there is the value of the scalar field on the horizon, with $h(r_h) = 0$ reached as $r_h \to 0$ and $h(r_h)$ very close to one for large values of the the event horizon radius; also, we have normalized $T_H M$ and $T_H^2 A_H$ such that the unit value is approached for a Schwarzschild black hole.

The picture is different for $\alpha_c < \alpha < \alpha_{max}$ (for example for $\alpha = 1.35$ in Figure 4), in which case we notice the following pattern. As r_h increases, the metric function $N(r)$ starts to develop a second minimum, and, for some critical value of r_h, one finds $N(r_c) =$

$N'(r_c) = 0$, with $r_c > r_h$ depending on α. Therefore an extremal horizon occurs in the outside region, which is the limiting configuration of this set of solutions (thus the range of r_h is bounded for $\alpha_c < \alpha < \alpha_{max}$). This horizon is regular, as found by computing some curvature invariants, with both $\sigma(r_c)$ and $h(r_c)$ taking values different from one there. Thus these black holes develop an infinite throat and become gravitationally closed.

It is instructive to consider also a fixed value of r_h and to vary the value of α. As one can see in Figure 5, the solutions with $\alpha = 0$ correspond in this case to "regularised" global monopoles in a fixed Schwarzschild background (*e.g.* one finds $8\pi T_H M \to 1$ as $\alpha \to 0$). As α increases, the geometry of the spacetime deviates from the Schwarzschild one. Then, for a critical value of α, the metric function $N(r)$, apart from a simple zero at $r = r_h$, develops again a double zero at some $r_c > r_h$. Again, a similar picture is found for black holes with large enough values of α in the gravitating Georgi-Glashow model, see *e.g.* the review work [3].

3.3 Extremal black holes. An $\mathbf{AdS}_2 \times S^2$ exact solution

The near horizon expansion of the extremal black holes is more constrained, since the metric function $N(r)$ has a double zero at the horizon, $N(r) = N_2(r - r_h)^2 + O(r - r_h)^3$, while the expansion for $\sigma(r)$ and $h(r)$ is still given by (22), with

$$N_2 = \frac{(1-h_0^2)(1-3h_0^2)^2}{h_0^2(1+h_0^2)}, \quad \sigma_1 = \frac{4(1+h_0^2)(5h_0^2-1)\left((h_0^2-1)(1-3h_0^2)\right)^{5/2}}{h_0(3-7h_0^2+17h_0^4-21h_0^6)^2}\sigma_h, \quad (28)$$
$$h_1 = \frac{2(1+h_0^2)\left((h_0^2-1)(1-3h_0^2)\right)^{3/2}}{-3+7h_0^2-17h_0^4+21h_0^6}.$$

Moreover, the event horizon radius of an extremal configuration is also fixed, $r_h = \frac{h_0}{\sqrt{(h_0^2-1)(1-3h_0^2)}}$. The parameter $h_0 = h(r_h)$ is determined by the coupling constant α as a solution of the equation $\sqrt{2}h_0\sqrt{1-h_0^4} = 1/\alpha$.

The occurrence of extremal black holes in our model is also suggested by the existence of the following exact solution of the Einstein-Goldstone equations (3), (5), corresponding to an $AdS_2 \times S^2$ spacetime with a constant scalar field magnitude:

$$ds^2 = v_1\left(\frac{dr^2}{r^2} - r^2dt^2\right) + v_2(d\theta^2 + \sin^2\theta d\varphi^2), \quad h(r) = h_0, \quad (29)$$

where

$$v_1 = \frac{h_0^2(1+h_0^2)}{(3h_0^2-1)^2(1-h_0^2)}, \quad v_2 = \frac{h_0^2}{(3h_0^2-1)(1-h_0^2)}, \quad \alpha = \frac{1}{\sqrt{2}h_0\sqrt{1-h_0^4}}. \quad (30)$$

As seen from (28), this solution describes the neighborhood of the event horizon of an extremal black hole. (The far field expression of an extremal solution is still given by (27).) The configuration (29) provides also an analytical explanation for some features revealed in the numerical analysis. For example, one can see that the range of h_0 is restricted,

$1/\sqrt{3} < h_0 < 1$, while $3\sqrt{3}/4 < \alpha < \infty$. (Also, as h_0 approaches the limiting values, the sizes of the AdS_2 and S^2 parts of the metric diverge.) Then it is clear that extremal black holes may exist for $\alpha > 3\sqrt{3}/4$ only. It would be interesting to consider these solutions in the context of the attractor mechanism and to compute their entropy function.

One can verify that no $AdS_2 \times S^2$ solutions are found in the global monopole model (18), which is consistent with the absence of extremal black holes for the numerical solutions discussed in [8], [9].

4 Further remarks

The main purpose of this work was to present finite mass black hole and soliton solutions in a theory with scalar fields featuring a spontaneously broken symmetry. To this end we have used a specific Goldstone model described by an $O(3)$ isovector scalar field, originally proposed in [14].

Not entirely surprisingly, it turns out that a number of basic feature of our solutions are rather similar to those of the well-known gravitating gauged monopoles, and not to the usual global monopoles solutions of the model (18). What the present (global-Goldstone) monopole has in common with the local gauged ('t Hooft-Polyakov) monopole is the topological charge and lower bound, both of which are absent in the usual global monopole. In both cases the topology is encoded in the scalar iso-multiplet. This may indicate that, also in the gravitating case, some basic properties of the gauged monopole can be attributed to the scalar fields.

That our gravitating Goldtsone solutions are akin to gravitating 't Hooft-Polyakov monopoles [12], and distinct from gravitating Skyrmions [4] is expected. This is because like the former [12], our solutions decay as *monopoles* unlike the Skyrmions which decay like instantons, *i.e.* at a faster rate as *pure gauge*. A very simple manifestation of this feature is the different behaviours of the radial functions $h(r)$ and $w(r)$ describing (a) the Higgs field of a spherically symmetric monopole, and respectively (b) the gauge field of a spherically symmetric instanton. The boundary values of these functions between $[r = 0\,,\, r = \infty]$ are $[h(0) = 0\,,\, h(\infty) = 1]$ and $[w(0) = \pm 1\,,\, w(\infty) = \mp 1]$ respectively. It is also known [24] that the dynamics of the unit charge soliton of a $O(D+1)$ (Skyrme) sigma model on $\mathbb{R}^D$, described by the *chiral* function $f(r)$ is identical to that of the corresponding Yang-Mills (YM) instanton form factor $w(r)$, *via*, $w(r) = \cos f$. Thus, the boundary values of the chiral function $f(r)$ of a Skyrmion, $[f(0) = \pi \,\mathrm{or}\, 0\,,\, f(\infty) = 0 \,\mathrm{or}\, \pi]$, are instantonic. It is therefore not surprising that with gravitating Goldstone solutions we encounter extremal black holes, which in the case of gravitating Skyrmions these are absent, just as the case is for the Einstein-YM (sphaleron) black holes in [25].

Having said this however, the Skyrmion is a topologically stable soliton like the Goldstone soliton, and unlike the YM sphaleron. In this respect, the gravitating Skyrmion is more akin to our Goldstone solutions and different from the Bartnik-McKinnon solution [26] of the Einstein-YM model, as shown to be stable in [5]. Also, although the dominant energy conditions is satisfied by the model in this work, similar to the Skyrme theory, it involves nonrenormalizable interactions.

Also, similar to the case of Einstein-Skyrme theory, we could show that the specific gravitating Goldtsone model in this work has also black hole solutions stable against linear fluctuations. In examining time-dependent fluctuations around the solutions in Section 3, all field variables are written as the sum of the static equilibrium solution whose stability we are investigating and a time dependent perturbation. By following the standard methods, we derive linearized equations for $\delta\sigma(r,t)$, $\delta N(r,t)$ and $\delta h(r,t)$. The linearized equations imply that $\delta\sigma(r,t)$, $\delta N(r,t)$ are determined by $\delta h(r,t)$. For an harmonic time dependence $e^{-i\Omega t}$, the linearized system of the matter sector implies a standard Schrödinger equation

$$\left\{-\frac{d^2}{d\rho^2}+U(\rho)\right\}\beta(\rho)=\Omega^2\beta(\rho), \tag{31}$$

where $\beta=\delta h/g$ (with g a strictly positive function of the unperturbed variables) and a new radial coordinate is introduced, $d/d\rho=N\sigma d/dr$. The expression of the potential is very complicated and we shall not give it here. However, $U(\rho)$ is regular everywhere, with $U(r_h)=0$ and $U(\infty)=4$. It follows that Eq. (31) will have no bound states if the potential $U(\rho)$ is everywhere greater than the lower of its two asymptotic values *i.e.* $U(\rho)>0$. Indeed, this condition was satisfied by some of our black hole solutions.

Perhaps the most unusual feature of the model in this work is the existence of extremal black hole configurations, even in the absence of gauge fields. Moreover, by using the approach in [27], [28] one can show the first law of thermodynamics for our black hole solutions reads $dM=T_H dS$. Thus there is no work term associated with the scalar field, which partially can be attributed to the $1/r^2$ decay of the scalar field ϕ at infinity, see (27). (Note that the scalar charge Σ, as defined from the large r expression $\phi\sim\phi_\infty+\Sigma/r$, may enter the first law [29]). The same form of the first law is found for black holes with Skyrme hair [27], [30]; however, no extremal configurations exist in that case. The existence of extremal black holes for the model in this work, viewed as a balance of two different charges, may look puzzling, since the only charge we can define on the scalar sector is a topological one. However, this can be understood in analogy with the magnetically charged Reissner-Nordström black hole embedded in a non-Abelian theory. In that case, the magnetic charge is quantised and, accordingly, there is no work term associated with it in the first law, despite the existence of extremal black holes. Similarly, the solutions considered in this work have unit 'magnetic' charge, which has a topological origin and likewise cannot enter the first law.

As a direction for future work, it would be interesting to study the case of solutions with a nonvanishing scalar symmetry-breaking potential. In the presence of gauge fields, the inclusion of this term is known to change some properties of gravitating monopoles [12] drastically. Here, the same would be expected of the gravitating Goldstone, since that is also a monopole theory.

Also, working in a flat spacetime background, Ref. [21] has given numerical evidence for the existence of axially symmetric generalisations of the spherically symmetric solutions of the model in this work. These are multisolitons with topological charge $n>1$, and unstable soliton-antisoliton pairs with zero topological charge. By employing methods similar to those used in the study of Einstein-Yang-Mills-Higgs multi-monopoles [31], we could confirm the existence of gravitating generalisations of the axially symmetric configurations in [21]. This includes also black hole solutions whose horizon has a spherical topology, but geometrically differs from a sphere. However, due to the highly nonlinear nature of the problem, we could not clarify the issue of the limiting behaviour of these gravitating axially symmetric configurations.

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